Fermions on the brane in 6D with nonsingular exponential scale factors

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We introduce new realistic brane-solutions with exponential scale factors in the $6D$-space-time. We show that for these solutions the zero modes of all bulk fields are sharply localized at different positions on the brane and have "Gaussian shape" wave-functions in the extra space. We also explicitly show that in the model there are cases when exactly three fermion generations naturally arise only through gravity. Because of localized fermion modes are also stuck at different positions in the extra space, there is possibility to provide a framework for natural explaining the fermion mass hierarchy in terms of higher dimensional geography.

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I. INTRODUCTION

The idea of extra dimensions [1, 2, 3, 4, 5, 6] is one of the most attractive ideas concerning unification of gauge fields with general relativity and new solutions to old problems (smallness of cosmological constant, the origin of the hierarchy problem, the nature of flavor, etc.). In theories with extra dimensions our world is associated with a brane, embedded in a higher-dimensional space-time with non-compact extra dimensions and non-factorizable geometry. The key ingredient for realizing the brane world idea is localization of various bulk fields on a brane by a natural mechanism. The gravity is known to be the unique interaction having universal coupling with all matter fields, so it is important to find a purely gravitational trapping mechanism. The brane solutions and matter localization mechanisms has been widely investigated in scientific literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Recently the gravitational trapping of the zero modes of all bulk fields was realized for the brane solutions with non-exponential warp factors in 6-dimensional bulk space-time [25, 26, 27]. In our previous article [28] in the case of the non-exponential scale factors we also have suggested new purely gravitational mechanism explaining the origin of three generations of the Standard Model fermions and explicitly show that localized fermions are stuck at different points on the brane in the extra space. In these models, with non-exponential scale factors, the zero-mode solutions of the bulk fields are normalizable and integrals of there Lagrangians over the extra coordinates are finite, but their wave functions spread rather widely in the bulk owing to the lack of the non-exponential warp factor. Thus, in order not to contradict the strict experiments such as the charge conservation law, some parameters in this models must be chosen in a proper way. In some cases it is hard to say whether there is such a suitable choice of the parameters. From this point of view it is interesting to consider the models with exponential warp factors when the wave functions of localized fields are sharply peaked on the brane in the extra space. At present it is known (see for example [12, 17]) that in $5D$ and $6D$ models with exponential warp factors it is possible to localize all spin fields if one introduces non-gravitational interactions. So it is of great interest to realize a purely gravitational localization mechanism in the models with exponential scale factors.

In this article in the case of $(1 + 5)$-space-time we introduce new brane-solutions with exponential scale factors sharply localizing the zero modes of all local fields (spin 0 scalar field, spin $^1\!/_2$ spinor field, spin 1 gauge field, spin $^3\!/_2$ gravitino field, spin 2 gravitational field and totally antisymmetric tensor fields) on the brane in the extra space, and for the realistic physical situation of matter distribution in the $6D$-space-time we realize the purely gravitational trapping mechanism explaining the origin of three generations of the Standard Model fermions from one generation in a higher-dimensional theory and explicitly show that fermions are sharply stuck at different points on the brane in the extra space.

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II.   MAIN NOTATIONS AND EQUATIONS

The action and Einstein's equations which we consider in this article have the form:

$$S = \int d^6 x \sqrt{-g} \left[\frac{M^4}{2} \left(R + 2\Lambda \right) + L \right] , \quad R_{AB} - \frac{1}{2} g_{AB} R = \frac{1}{M^4} \left(\Lambda g_{AB} + T_{AB} \right) , \tag{1}$$

where M, R, Λ, L, R_{AB} and T_{AB} are respectively the fundamental scale, the scalar curvature, the cosmological constant, the Lagrangian of matter fields, the Ricci and the energy-momentum tensors. In the bulk space-time we use the following metric ansatz [24, 25, 26, 27, 28] : $ds^2 = \phi^2 (r) g_{\alpha\beta} dx^\alpha dx^\beta - g(r) \left(dr^2 + r^2 d\theta^2 \right)$. The source of the brane is described by a stress-energy tensor T_{AB}, its nonzero components we choose in the form $T_{\alpha\beta} = -g_{\alpha\beta} F_0 (r)$, $T_{ij} = -g_{ij} F(r)$, where F_0 and F are source functions, which depend only on the radial coordinate r in the extra space. In this case the Einstein's equations take the following form

$$\frac{3\phi''}{\phi} + \frac{3\phi'^2}{\phi^2} + \frac{3\phi'}{r\phi} + \frac{g''}{2g} - \frac{g'^2}{2g^2} + \frac{g'}{2rg} = \frac{g(F_0 - \Lambda)}{M^4} + \frac{g\Lambda_{phys}}{\phi^2 M_P^2}, \tag{2}$$

$$\frac{6\phi'^2}{\phi^2} + \frac{2\phi' g'}{\phi g} + \frac{4\phi'}{r\phi} = \frac{g(F - \Lambda)}{M^4} + \frac{2g\Lambda_{phys}}{\phi^2 M_P^2}, \quad \frac{4\phi''}{\phi} + \frac{6\phi'^2}{\phi^2} - \frac{2\phi' g'}{\phi g} = \frac{g(F - \Lambda)}{M^4} + \frac{2g\Lambda_{phys}}{\phi^2 M_P^2}, \tag{3}$$

where the prime denotes differentiation d/dr. The constant Λ_{phys} represents the physical four-dimensional cosmological constant, where $R_{\alpha\beta}^{(4)} - \frac{1}{2} g_{\alpha\beta} R^{(4)} = \frac{\Lambda_{phys}}{M_P^2} g_{\alpha\beta}$. In this equation $R_{\alpha\beta}^{(4)}$, $R^{(4)}$ and M_P are four-dimensional physical quantities: Ricci tensor, scalar curvature and Planck scale.

III.   NONSINGULAR BRANE SOLUTION WITH EXPONENTIAL SCALE FACTORS

In the case $\Lambda_{phys} = 0$ from the equations (2)-(3) we can find [22, 23, 28] : $F' + 4\frac{\phi'}{\phi} (F - F_0) = 0$; $g = \frac{\delta\phi'}{r}$; $r\frac{\phi''}{\phi} + 3r\frac{\phi'^2}{\phi^2} + \frac{\phi'}{\phi} = \frac{rg}{2M^4} (F - \Lambda)$, where δ denotes the integration constant with the units of squared length. The possible two solutions to this equations are (in what follows we call these solutions "internal solutions" and introduce subscript "int"):

1. The first solution with scale factor ϕ_{int} monotonically decreasing from the value equal to 1 at the origin $z = 0$ in the extra space and asymptotically approaching the the nonzero positive value equal to $1 - d$ at the radial infinity $z = +\infty$

$$\phi_{\text{int}} (z) = 1 - d \left[1 - e^{-z^p} \right], \quad g_{\text{int}} (z) = \delta dp \varepsilon^{-2} z^{p-2} e^{-z^p}, \quad 0 < d < 1, \ p \geq 2, \ \delta > 0, \tag{4}$$

2. The second solution with scale factor ϕ_{int} monotonically increasing from the value equal to 1 at the origin $z = 0$ in the extra space and asymptotically approaching the positive value equal to $1 + d$ at the radial infinity $z = +\infty$

$$\phi_{\text{int}} (z) = 1 + d \left[1 - e^{-z^p} \right], \quad g_{\text{int}} (r) = \delta dp \varepsilon^{-2} z^{p-2} e^{-z^p}, \quad d > 0, \ p \geq 2, \ \delta > 0. \tag{5}$$

In (4)-(5) we have introduced the dimensionless radial coordinate $z = r/\varepsilon$, where ε is some positive constant with the units of length. In both cases the scale factor ϕ_{int} has an inflection point $r_{\text{infl}.\phi} = \varepsilon \left(\frac{p-1}{p} \right)^{\frac{1}{p}}$ (this value characterizes the order of the "effective width" Δ_{Brane} of the "thick brane" in the extra space). These solutions have equal scale functions $g_{\text{int}.}$. In the case $p = 2$ this function monotonically decreases from value equal to $\frac{2\delta d}{\varepsilon^2}$ at the origin $z = 0$ to the value equal to zero at the infinity $z = +\infty$ in the extra space and its derivative g'_{int} is equal to zero at the origin and at the infinity in the extra space. In the case $p > 2$ this function is equal to zero at the origin $r = 0$ in the extra space and monotonically increases in the range $0 < r < r_{\max.g}$, approaches the maximum value at the $r_{\max.g}$ and then monotonically decreases in the range $r_{\max.g} < r < +\infty$ and approaches the asymptotic value equal to zero at the radial infinity, where $r_{\max.g} = \varepsilon \left(\frac{p-2}{p} \right)^{\frac{1}{p}}$. It must be mentioned that the scale factor $g_{\text{int}}(r)$ defines the topology of the brane. These solutions exist for any Λ and corresponding source functions F_0 and F can be easily calculated from

corresponding Einstein's equations using (4) and (5): $F = \Lambda - \frac{2pM^4}{\delta}\phi^{-2}\left[(4\phi - 3c)\ln\left(\frac{\phi-c}{1-c}\right) + \phi\right]$, $F_0 = F + \frac{\phi}{4}\frac{dF}{d\phi}$, where $c = 1 - d$.

Similarly, as it was done in our article [28] for the case of non-exponential scale factors, it is easy to explicitly show that these solutions localize the zero modes of the following bulk local fields: spin 0 scalar field, spin 1 gauge field, spin $3/2$ gravitino field, spin 2 gravitational field and totally antisymmetric tensor fields. The zero modes of these fields will be localized at different positions on the brane in the extra space, but as against a former case [28], in this case we shall have sharp localization of the modes with "Gaussian shapes" of wave-functions in the extra space due to exponential type of scale factors. Here we want to consider in more details the localization of the bulk spin-$1/2$ fermion in the background solution (4) (for the solution (5) the results are the same). Of course, in due analysis, in what follows we will neglect the back-reaction on the geometry induced by the existence of the bulk fields, and, without loss of generality, we will take a flat metric on the brane. Fermion action and corresponding equation of motion are

$$S_{\frac{1}{2}} = \int d^6x\sqrt{-^6g}\,\overline{\Psi}i\Gamma^M D_M\Psi \ ; \quad \Gamma^M D_M\Psi = 0 \ . \tag{6}$$

Introducing the vielbein $h_A^{\widetilde{A}}$ through the usual definition $g_{AB} = h_A^{\widetilde{A}}h_B^{\widetilde{B}}\eta_{\widetilde{A}\widetilde{B}}$ where $\widetilde{A}, \widetilde{B}, ...$ denote the local Lorentz indices, it is easy to find the non-vanishing components of the spin-connection for the background (4): $\omega_\alpha^{\widetilde{z}\widetilde{\alpha}} = \frac{\phi'}{\varepsilon\sqrt{g}}$, $\omega_\theta^{\widetilde{z}\widetilde{\theta}} = \frac{\partial_z(z\sqrt{g})}{\sqrt{g}}$. Therefore the covariant derivatives have form $D_\alpha\Psi = \left(\partial_\alpha + \frac{1}{2}\omega_\alpha^{\widetilde{z}\widetilde{\alpha}}\gamma_z\gamma_\alpha\right)$, $D_z\Psi = \partial_z\Psi$, $D_\theta\Psi = \left(\partial_\theta + \frac{1}{2}\omega_\theta^{\widetilde{z}\widetilde{\theta}}\gamma_z\gamma_\theta\right)$. The 6-dimensional spinor $\Psi\left(x^M\right)$ can be decomposed on a 4-component spinor $\psi\left(x^\mu\right)$ and 2-component spinor $\xi\left(z,\theta\right)$: $\Psi\left(x^M\right) = \psi\left(x^\mu\right)\otimes\xi\left(z,\theta\right)$. We require that the 4-dimensional spinor satisfies the equation $\gamma^\mu\partial_\mu\psi\left(x^\beta\right) = m\psi\left(x^\beta\right)$. The 2-component spinor can can be expanded over the states with fixed rotation momentum l in the θ direction: $\xi\left(z,\theta\right) = \sum\xi_l\left(z\right)e^{il\theta}$, where $\xi_l\left(z\right) = \left(\upsilon_l, \zeta_l\right)$ is 2-component spinor, υ_l and ζ_l are functions of variable z. Taking gamma matrices γ^z and γ^θ in the form $\gamma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^\theta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, as a result we obtain the following equations for υ_l and ζ_l

$$\frac{m\varepsilon}{\phi}\upsilon_l + \frac{1}{\sqrt{g_{\text{int}}}}\left[\partial_z + 2\frac{\phi'_{\text{int}}}{\phi_{\text{int}}} + \frac{1}{2}\frac{\partial_z\left(zg_{\text{int}}^{\frac{1}{2}}\right)}{zg_{\text{int}}^{\frac{1}{2}}} + \frac{l}{z}\right]\zeta_l = 0 \ ,$$

$$\frac{m\varepsilon}{\phi}\zeta_l + \frac{1}{\sqrt{g_{\text{int}}}}\left[\partial_z + 2\frac{\phi'_{\text{int}}}{\phi_{\text{int}}} + \frac{1}{2}\frac{\partial_z\left(zg_{\text{int}}^{\frac{1}{2}}\right)}{zg_{\text{int}}^{\frac{1}{2}}} - \frac{l}{z}\right]\upsilon_l = 0 \ . \tag{7}$$

In the zero-mass mode case $m = 0$ (the case $m \neq 0$ we will consider later on) the solution to these equations reads $\upsilon_l\left(z\right) = a_l\phi_{\text{int}}^{-2}g_{\text{int}}^{-\frac{1}{4}}z^{-\frac{1}{2}+l}$, $\zeta_l\left(z\right) = b_l\phi_{\text{int}}^{-2}g_{\text{int}}^{-\frac{1}{4}}z^{-\frac{1}{2}-l}$, $\xi_l\left(z\right) = \phi_{\text{int}}^{-2}g_{\text{int}}^{-\frac{1}{4}}z^{-\frac{1}{2}}\begin{pmatrix} a_l z^l \\ b_l z^{-l} \end{pmatrix}$, with a_l and b_l being the integration constants. Using explicit form (4) of the scale functions ϕ_{int} and g_{int} , the normalization requirement becomes $1 = 2\pi\left(\delta dp\varepsilon^2\right)^{\frac{1}{2}}\int\limits_0^{+\infty}\left[a_l^2 z^{2l} + b_l^2 z^{-2l}\right]z^{\frac{1}{2}p-1}e^{-\frac{1}{2}z^p}dz$, and it is easy to see that the extra part of the $6D$-wavefunction is normalizable (i.e., the integral over z converges) only in the following cases: I) $a_l \neq 0$, $b_l = 0$, $l > -p/4$; II) $a_l = 0$, $b_l \neq 0$, $l < p/4$; III) $a_l \neq 0$, $b_l \neq 0$, $-p/4 < l < p/4$. Finally for the effective wave function of the zero-mass mode of 6-dimensional fermion in flat space we have $\Psi = \psi\left(x^\mu\right)\otimes\left(\delta dp\varepsilon^2\right)^{\frac{1}{4}}z^{\frac{1}{4}p-\frac{1}{2}}e^{-\frac{1}{4}z^p}\left[\sum\limits_{-\frac{p}{4}<l<\frac{p}{4}}\begin{pmatrix} a_l z^l e^{il\theta} \\ b_l z^{-l} e^{il\theta} \end{pmatrix} + \sum\limits_{l\geq\frac{p}{4}}\begin{pmatrix} a_l z^l e^{il\theta} \\ b_{-l} z^l e^{-il\theta} \end{pmatrix}\right]$. In this case because of exponential decreasing warp factors normalizability of wave function means sharp localization of corresponding mode on the brane. These components of zero-mass mode of 6-dimensional bulk fermion are localized at different points $r_{\text{max.},l} = \varepsilon\left(\frac{p+4l-2}{p}\right)^{\frac{1}{p}}$ in the extra space. Each component can be regarded as 4-dimensional massless fermion. So in this case there are infinite number of massless fermions on the brane. This situation indicates that in order to single out finite number of these localized fermion components we need some new mechanism, and the one possible way we will consider in the next section.

To conclude this section let us consider the solution to Einstein's equations without any sources ($F_0\left(r\right) \equiv F\left(r\right) \equiv 0$) in the case $\Lambda \neq 0$, $\Lambda_{phys} = 0$. In this case the Einstein's equations reduce to $r\frac{\phi''}{\phi} + 3r\frac{\phi'^2}{\phi^2} + \frac{\phi'}{\phi} = -\frac{rg}{2M^4}\Lambda$, $g = \frac{\delta\phi'}{r}$, $\delta = const$. The solution to these equations is (in what follows we call it "external solution" and introduce

corresponding subscript "ext")

$$\phi_{\text{ext}}(z) = \frac{1}{a + b\ln z}, \quad g_{\text{ext}}(z) = \frac{\delta b}{\varepsilon^2}\frac{1}{z^2\left[a+b\ln z\right]^2}, \quad b = \frac{\delta\Lambda}{10M^4}, \tag{8}$$

where the a is some new positive parameter and the constants ε, δ and the radial coordinate z are the same as above. This solution is nonsingular in the range $e^{-\frac{a}{b}} < z < +\infty$ in the extra space. Its scale factors $\phi_{\text{ext.}}$ and $g_{\text{ext.}}$ are monotonically decreasing functions in this range.

IV. THREE FERMION GENERATIONS ON THE BRANE

Now imagine the following realistic physical situation. Suppose in the extra space in the range $0 \le z \le z_0$ (in what follows we name this area in the extra space by core) we have nonzero source functions $F_0(z)$ and $F(z)$ which correspond to the solution (4) with decreasing scale factor $\phi_{\text{int.}}$, and out of the core, i.e. in the range $z > z_0$ there are no source functions, i.e. $F_0(z) \equiv F(z) \equiv 0$ if $z > z_0$. At present we do not argue the origin of such matter distribution in the $6D$-space-time. Let us denote by ϕ and g the scale factors of solution to Einstein's equations in this special case. It is obvious that they can be constructed from the scale factors of solutions (4) and (8) in these two regions in the following way: $\phi(z) = \begin{bmatrix} \phi_{\text{int}}(z), & 0 \le z \le z_0, \\ \phi_{\text{ext}}(z), & z > z_0, \end{bmatrix}$; $g(z) = \begin{bmatrix} g_{\text{int}}(z), & 0 \le z \le z_0, \\ g_{\text{ext}}(z), & z > z_0. \end{bmatrix}$ To avoid singularities we impose on the scale factors and there first derivatives the continuity conditions at the boundary of the core $z = z_0$: $\phi_{\text{int}}(z_0) = \phi_{\text{ext}}(z_0)$; $\phi'_{\text{int}}(z_0) = \phi'_{\text{ext}}(z_0)$; $g_{\text{int}}(z_0) = g_{\text{ext}}(z_0)$; $g'_{\text{int}}(z_0) = g'_{\text{ext}}(z_0)$. So far there were the following free parameters in the model: $M, \Lambda, \delta, d, \varepsilon, p, a, z_0$. Solving the continuity conditions we can fix the following parameters $a = \frac{2-(x_0-1)\ln x_0}{4x_0}\lambda(x_0)$, $b = \frac{\delta\Lambda}{10M^4} = \frac{x_0-1}{4x_0}p\lambda(x_0)$, $d = \frac{(x_0-1)e^{x_0}}{\lambda(x_0)}$, where we have denoted $x_0 = z_0^p > 1$ and $\lambda(x) = (x-1)e^x + x + 1$. Now we have following free parameters in the model: $M, \delta, \varepsilon, p$ and $z_0 > 1$.

Let turn our attention to the localization of the bulk spin-$\frac{1}{2}$ fermions on the brane described by our solution. In this case equations for the components υ_l and ζ_l of the radial part $\xi_l(z)$ of zero-mass wave-function have the same form as in the previous section, and the solution reads

$$\xi_l(z) = \phi^{-2}g^{-\frac{1}{4}}z^{-\frac{1}{2}}\begin{pmatrix} a_l z^l \\ b_l z^{-l} \end{pmatrix} = \begin{bmatrix} \phi_{\text{int}}^{-2}g_{\text{int}}^{-\frac{1}{4}}z^{-\frac{1}{2}}\begin{pmatrix} a_l z^l \\ b_l z^{-l} \end{pmatrix}, & 0 \le z \le z_0, \\ \phi_{\text{ext}}^{-2}g_{\text{ext}}^{-\frac{1}{4}}z^{-\frac{1}{2}}\begin{pmatrix} a_l z^l \\ b_l z^{-l} \end{pmatrix}, & z > z_0, \end{bmatrix} \tag{9}$$

with the following normalization requirement

$$1 = 2\pi\varepsilon\left[\sqrt{\delta dp}\int_0^{z_0} z^{\frac{p}{2}-1}e^{-\frac{z^p}{2}}\left(a_l^2 z^{2l} + b_l^2 z^{-2l}\right)dz + \sqrt{\delta b}\int_{z_0}^{+\infty}\frac{a_l^2 z^{2l} + b_l^2 z^{-2l}}{z(a+b\ln z)}dz\right]. \tag{10}$$

Analysis of convergence of the last integral at the boundaries of the whole integration domain (i.e., at the points $z = 0$ and $z = +\infty$) shows that normalizable components of the zero-mass mode exist only in the following cases: I) $a_l \ne 0$, $b_l = 0$, $-p/4 < l < 0$; II) $a_l = 0$, $b_l \ne 0$, $0 < l < p/4$. So the number of normalizable zero-mass modes depends on the value of parameter p. Namely in the case $12 < p \le 16$ we have exactly three normalizable zero-mass solutions with $l = \pm 1, \pm 2, \pm 3$. In this case the effective wave-functions of this zero-mass modes in flat space are:

$$\Psi(x^M) = \begin{bmatrix} \psi(x^\mu) \otimes \left(\delta dp\varepsilon^2\right)^{\frac{1}{4}}z^{\frac{p}{2}-1}e^{-\frac{z^p}{4}}\sum_{0<l<\frac{p}{4}}\begin{pmatrix} a_{-l}z^{-l}e^{-il\theta} \\ b_l z^{-l}e^{il\theta} \end{pmatrix}, 0 \le z \le z_0, \\ \psi(x^\mu) \otimes \left(\delta b\varepsilon^2\right)^{\frac{1}{4}}z^{-1}(a+b\ln z)^{-1}\sum_{0<l<\frac{p}{4}}\begin{pmatrix} a_{-l}z^{-l}e^{-il\theta} \\ b_l z^{-l}e^{il\theta} \end{pmatrix}, z > z_0. \end{bmatrix} \tag{11}$$

They are localized in the radial direction of the extra space at the following points $r_{\text{ferm},l=1} = \varepsilon(1-6/p)^{1/p}$, $r_{\text{ferm},l=2} = \varepsilon(1-10/p)^{1/p}$, $r_{\text{ferm},l=3} = \varepsilon(1-14/p)^{1/p}$. It must be mentioned that if $12 < b < 14$ the last formula does not work, but in this subcase directly from the wave function (11) we have $r_{\text{ferm},l=3} = 0$. For the boundary of the core we have $z_0 > 1$, so each of these three modes are sharply localized inside the core and the wave-functions of these localized modes have the "Gaussian shapes" in the extra space. These three components of a higher dimensional fermion (stuck at different points on the brane in the extra space) can be identified with different 4-dimensional fermion generations, and so we have purely gravitational mechanism explaining the origin of three generations of the Standard Model fermions.

V. NON-ZERO-MASS MODES

Now let us consider the case $m \neq 0$ for the physical situation described in the previous section. We have to solve the equations (7) in two regions - inside the core and outside the core. We will seek the solution in the form $v_l(z) = \phi^{-2} g^{-\frac{1}{4}} z^{l-\frac{1}{2}} A(z)$, $\zeta_l(z) = \phi^{-2} g^{-\frac{1}{4}} z^{-l-\frac{1}{2}} B(z)$, where $\phi = \phi_{\text{int}}$ and $g = g_{\text{int}}$ if $0 \leq z \leq z_0$, and $\phi = \phi_{\text{ext}}$, $g = g_{\text{ext}}$ if $z > z_0$. From (7) for the functions A and B we get equations $\frac{d}{dz}\left[\frac{\phi}{\sqrt{g}} z^{2l} \frac{dA}{dz}\right] - m^2 \varepsilon^2 \frac{\sqrt{g}}{\phi} z^{2l} A = 0$, $\frac{d}{dz}\left[\frac{\phi}{\sqrt{g}} z^{-2l} \frac{dB}{dz}\right] - m^2 \varepsilon^2 \frac{\sqrt{g}}{\phi} z^{-2l} B = 0$. We examine the solution $A(z)$, the results for the $B(z)$ are the same. Taking into account (8) and normalization requirement for wavefunction at the radial infinity in the extra space, outside the core the equation for A_{ext} and its nomalizable solution assume the following form: $x^2 A''_{\text{ext}} + \left(\frac{2l}{p} + 1\right) x A'_{\text{ext}} - \frac{m^2 \delta b}{p^2} A_{\text{ext}} = 0$, $A_{\text{ext}}(x) = \alpha x^{-\frac{l+\sqrt{l^2 + m^2 \delta b}}{p}}$, where α is the integration constant and we have introduced new variable $x = z^p$. Inside the core the equation for A_{int} has the form

$$\frac{d}{dx}\left[\sigma(x) x^{\frac{4l+p}{2p}} \frac{dA_{\text{int}}}{dx}\right] - \frac{m^2 \delta d}{p} \frac{x^{\frac{4l-p}{2p}}}{\sigma(x)} A_{\text{int}} = 0, \quad \text{where} \quad \sigma(x) = \left[1 - d\left(1 - e^{-x}\right)\right] e^{\frac{x}{2}}. \tag{12}$$

At the origin of extra space on the solution we impose condition $|A_{\text{int}}(0)| < +\infty$, and at the boundary of the core $x = x_0$ we require the following matching conditions: $A_{\text{int}}(x_0) = A_{\text{ext}}(x_0)$, $A'_{\text{int}}(x_0) = A'_{\text{ext}}(x_0)$. So in the core, $0 \leq x \leq x_0$, we have a Sturm-Liouville differential equation for A_{int} with the following boundary conditions: $|A_{\text{int}}(0)| < +\infty$, $\frac{A'_{\text{int}}(x_0)}{A(x_0)} = -\frac{l+\sqrt{l^2 + m^2 \delta b}}{px_0}$. In the vicinity of the extra space origin for the asymptotic form of (12) and for its normalizable solution we have: $xA'' + \left(2\frac{l}{p} + \frac{1}{2}\right) A' - \frac{m^2 \delta d}{p} A = 0$, $A_{\text{int}}(x) = \beta x^{-\frac{4l-p}{4p}} BesselJ\left(\nu, \omega\sqrt{x}\right)$, where $\nu = \frac{4l-p}{2p}$, $\omega = 2\sqrt{\frac{m^2 \delta d}{p}}$ and β is the integration constant. To make some interesting qualitative conclusions we consider this solution as the correct solution on the entire core range and choose x_0 sufficiently large to use the asymptotic expression for the Bessel functions ($BesselJ(\nu, y)|_{y \to +\infty} \sim \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$) on the boundary of the core . Then for the eigenvalue Δ we have the following transcendental equation: $\tan\left(2\Delta - \frac{\pi l}{p}\right) = \Delta^{-1}\sqrt{\frac{l^2}{p^2} + \frac{b}{pdx_0}\Delta^2}$, $\Delta = \sqrt{\frac{m^2 \delta dx_0}{p}}$. This equation gives the discrete spectrum for $\Delta = \Delta_1, \Delta_2, \Delta_3, \dots$, and for corresponding nonzero masses we get: $m_n = \sqrt{\frac{p}{\delta dx_0}} \Delta_n$. Between zero and nonzero modes there is the gap depending on the value of parameter δ, so choosing the appropriate value of δ the non-zero modes can be neglected at low energies. The same conclusions can be done with respect to the exact solution of (12), although it is difficult to find the solutions explicitly.

VI. CONCLUSION

To conclude we summarize our main results. In this article in the case of $(1+5)$-space-time we have introduced new brane-solutions with exponential scale factors sharply localizing the zero modes of all local fields (spin 0 scalar field, spin $1/2$ spinor field, spin 1 gauge field, spin $3/2$ gravitino field, spin 2 gravitational field and totally antisymmetric tensor fields) on the brane in the extra space. We have explicitly shown that these solutions localize infinite number of $4D$-fermions on the brane. To solve this problem we have introduced another specific brane-solution with realistic source functions, i.e. the source functions F_0 and F are nonzero inside the core $0 \leq z \leq z_0$ in the extra space, and $F_0 \equiv F \equiv 0$ outside the core. We also have explicitly shown that in this model there are cases when exactly three $4D$-fermion generations can naturally arise only through gravity. As the zero modes of all other local bulk fields, these three zero modes of bulk fermion are sharply peaked at different positions on the brane and have "Gaussian shape" wave-functions in the extra space. So there is possibility to provide a framework for understanding the fermion mass hierarchy in terms of higher dimensional geography [29].

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